

Finite state automata

Data Structures and Algorithms for Computational Linguistics III
(ISCL-BA-07)

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Why study finite-state automata?

- Unlike some of the abstract machines we discussed, finite-state automata are efficient models of computation
- There are many applications
 - Electronic circuit design
 - Workflow management
 - Games
 - Pattern matching
 - ...

But more importantly ;-)

- Tokenization, stemming
- Morphological analysis
- Spell checking
- Shallow parsing/chunking
- ...

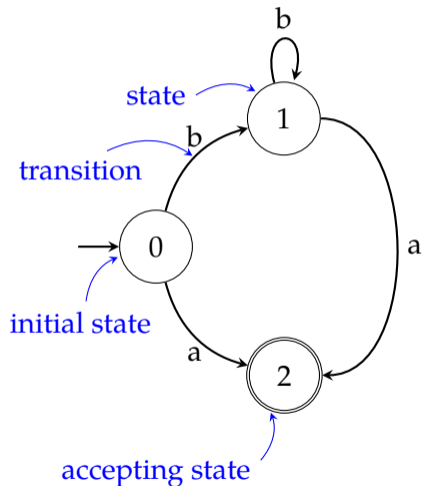
Finite-state automata (FSA)

- A finite-state machine is in one of a finite-number of states in a given time
- The machine changes its state based on its input
- Every regular language is generated/recognized by an FSA
- Every FSA generates/recognizes a regular language
- Two flavors:
 - *Deterministic finite automata* (DFA)
 - *Non-deterministic finite automata* (NFA)

Note: the NFA is a superset of DFA.

FSA as a graph

- An FSA is a directed graph
- States are represented as nodes
- Transitions are labeled edges
- One of the states is the *initial state*
- Some states are accepting states



DFA: formal definition

Formally, a finite state automaton, M , is a tuple $(\Sigma, Q, q_0, F, \Delta)$ with

Σ is the alphabet, a finite set of symbols

Q a finite set of states

q_0 is the start state, $q_0 \in Q$

F is the set of final states, $F \subseteq Q$

Δ is a function that takes a state and a symbol in the alphabet, and returns another state ($\Delta : Q \times \Sigma \rightarrow Q$)

At any state and for any input,
a DFA has a single well-defined action to take.

DFA: formal definition

an example

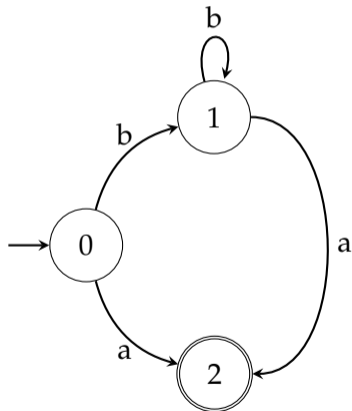
$$\Sigma = \{a, b\}$$

$$Q = \{q_0, q_1, q_2\}$$

$$q_0 = q_0$$

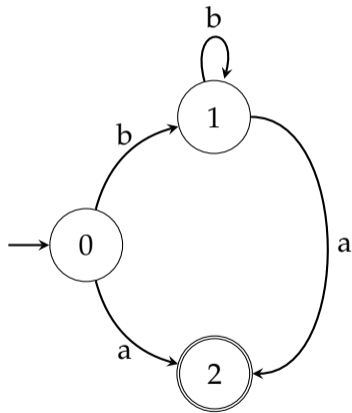
$$F = \{q_2\}$$

$$\Delta = \{(q_0, a) \rightarrow q_2, \quad (q_0, b) \rightarrow q_1, \\ (q_1, a) \rightarrow q_2, \quad (q_1, b) \rightarrow q_1\}$$



Another note on DFA

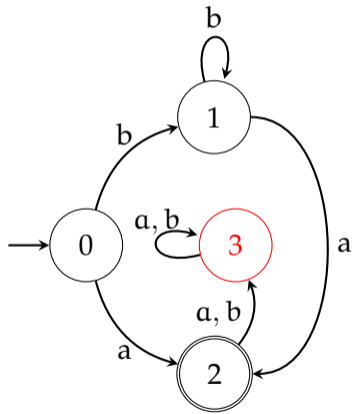
- Is this FSA deterministic?



Another note on DFA

error or sink state

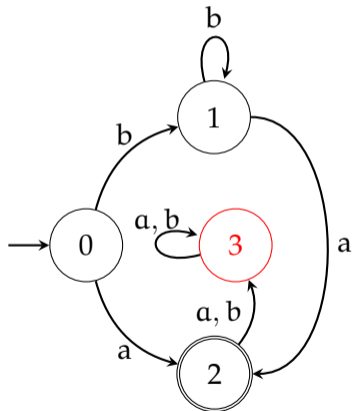
- Is this FSA deterministic?
- To make all transitions well-defined, we can add a sink (or error) state



Another note on DFA

error or sink state

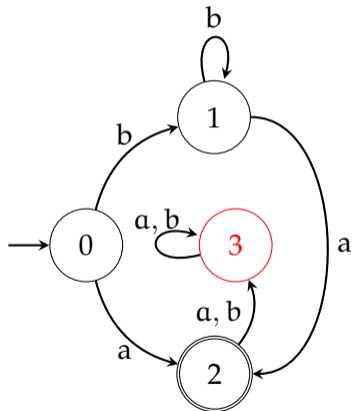
- Is this FSA deterministic?
- To make all transitions well-defined, we can add a sink (or error) state
- For brevity, we skip the explicit error state



Another note on DFA

error or sink state

- Is this FSA deterministic?
- To make all transitions well-defined, we can add a sink (or error) state
- For brevity, we skip the explicit error state
 - In that case, when we reach a dead end, recognition fails

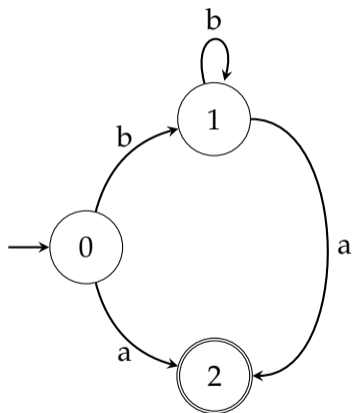


DFA: the transition table

transition table			
		<i>symbol</i>	
		a	b
<i>state</i>	→ 0	2	1
	1	2	1
	* 2	∅	∅

→ marks the start state

* marks the accepting state(s)

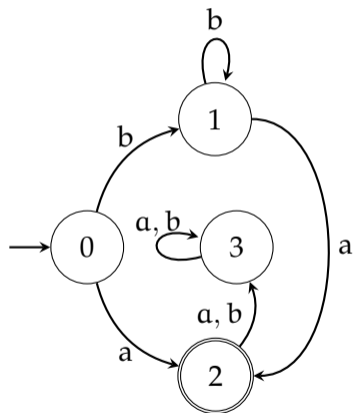


DFA: the transition table

transition table			
		<i>symbol</i>	
		a	b
<i>state</i>	→ 0	2	1
	1	2	1
	* 2	3	3
	3	3	3

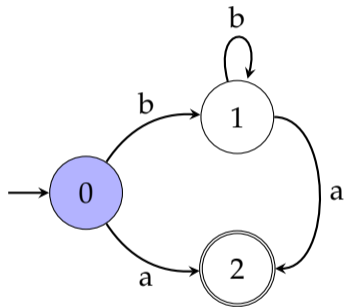
→ marks the start state

* marks the accepting state(s)



DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input

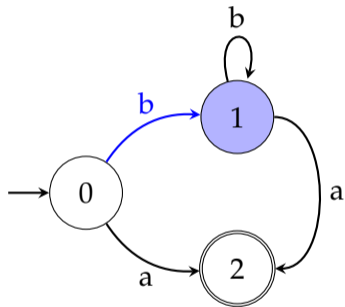


Input:

b	b	a
---	---	---

DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input

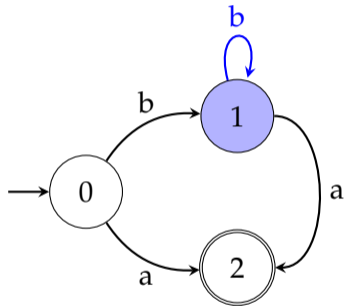


Input:

b	b	a
---	---	---

DFA recognition

1. Start at q_0
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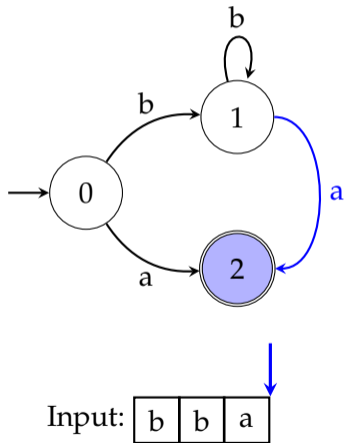


Input:

b	b	a
---	---	---

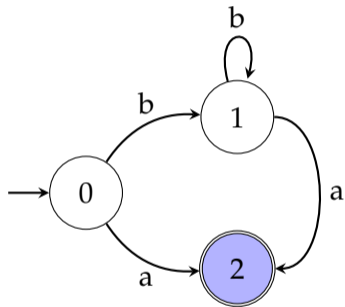
DFA recognition

1. Start at q_0
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DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input



Input:

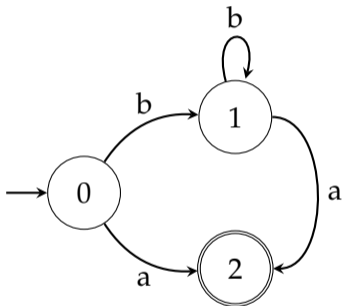
b	b	a
---	---	---

A blue arrow points to the 'a' in the input string.

DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input

- What is the complexity of the algorithm?
- How about inputs:
 - bbbb
 - aa

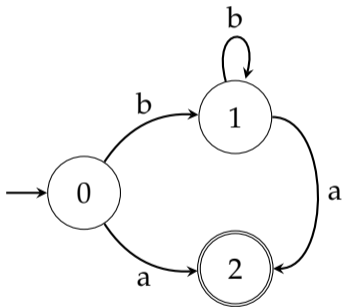


Input:

b	b	a
---	---	---

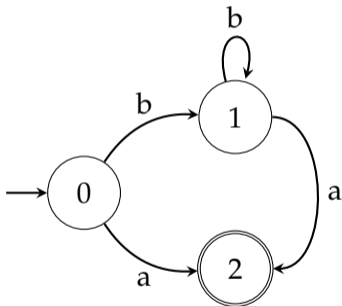
A few questions

- What is the language recognized by this FSA?



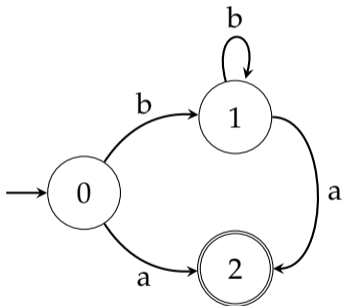
A few questions

- What is the language recognized by this FSA?
- Can you draw a simpler DFA for the same language?



A few questions

- What is the language recognized by this FSA?
- Can you draw a simpler DFA for the same language?
- Draw a DFA recognizing strings with even number of 'a's over $\Sigma = \{a, b\}$



Non-deterministic finite automata

Formal definition

A non-deterministic finite state automaton, M , is a tuple $(\Sigma, Q, q_0, F, \Delta)$ with

Σ is the alphabet, a finite set of symbols

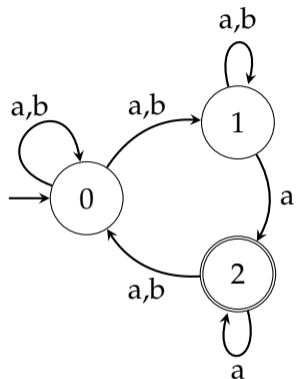
Q a finite set of states

q_0 is the start state, $q_0 \in Q$

F is the set of final states, $F \subseteq Q$

Δ is a function from (Q, Σ) to $P(Q)$, power set of Q ($\Delta : Q \times \Sigma \rightarrow P(Q)$)

An example NFA



transition table

		symbol	
		a	b
state	→0	0,1	0,1
	1	1,2	1
	*2	0,2	0

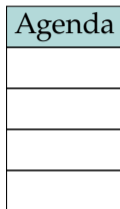
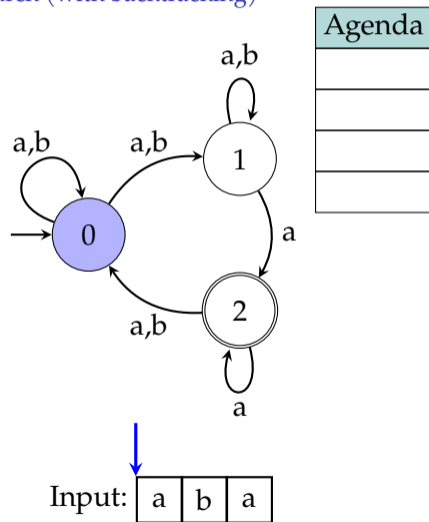
- We have nondeterminism, e.g., if the first input is a, we need to choose between states 0 or 1
- Transition table cells have *sets* of states

Dealing with non-determinism

- Follow one of the links, store alternatives, and *backtrack* on failure
- Follow all options in parallel

NFA recognition

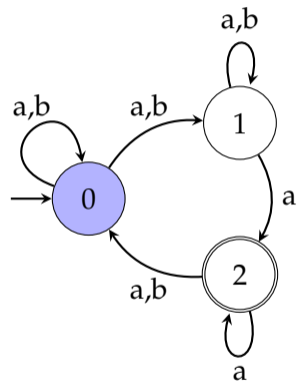
as search (with backtracking)



1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input
 - Accept if in an accepting state
 - Reject not in accepting state & agenda empty
 - Backtrack otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 1)$
$(q_1, 1)$

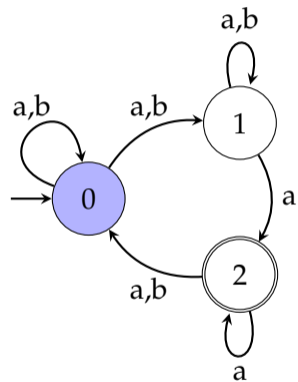
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Input:

a	b	a
---	---	---

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 1)$
$(q_1, 1)$

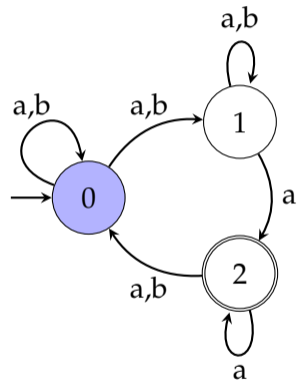
1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. **Get the next action from the agenda, act**
4. At the end of input
 Accept if in an accepting state
 Reject not in accepting state & agenda empty
 Backtrack otherwise

Input:

a	b	a
---	---	---

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 2)$
$(q_1, 2)$
$(q_1, 1)$

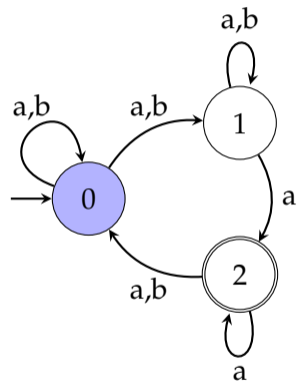
Input:

a	b	a
---	---	---

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input
 - Accept if in an accepting state
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 - Backtrack otherwise

NFA recognition

as search (with backtracking)



Agenda
(q ₀ , 2)
(q ₁ , 2)
(q ₁ , 1)

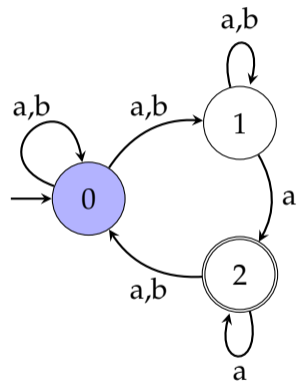
Input:

a	b	a
---	---	---

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. **Get the next action from the agenda, act**
4. At the end of input
 Accept if in an accepting state
 Reject not in accepting state & agenda empty
 Backtrack otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 3)$
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

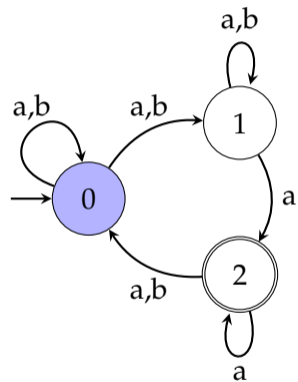
1. Start at q_0
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4. At the end of input
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 - Reject not in accepting state & agenda empty
 - Backtrack otherwise

Input:

a	b	a
---	---	---

NFA recognition

as search (with backtracking)



Agenda
(q ₀ , 3)
(q ₁ , 3)
(q ₁ , 2)
(q ₁ , 1)

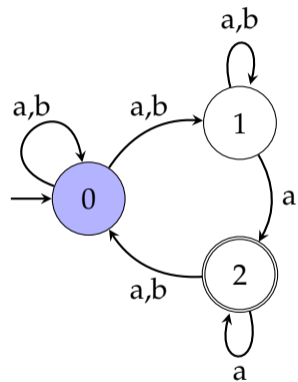
1. Start at q_0
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3. **Get the next action from the agenda, act**
4. At the end of input
 - Accept if in an accepting state
 - Reject not in accepting state & agenda empty
 - Backtrack otherwise

Input:

a	b	a
---	---	---

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input
 Accept if in an accepting state
 Reject not in accepting state & agenda empty

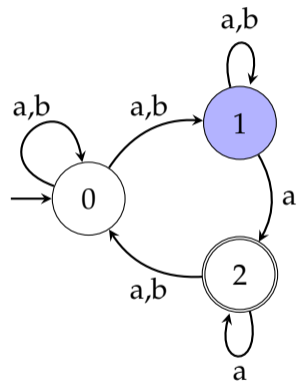
Backtrack otherwise

Input:

a	b	a
---	---	---

NFA recognition

as search (with backtracking)



Agenda
(q ₁ , 3)
(q ₁ , 2)
(q ₁ , 1)

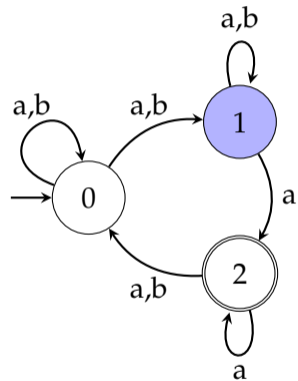
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Input:

a	b	a
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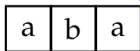
NFA recognition

as search (with backtracking)



Agenda
(q ₁ , 2)
(q ₁ , 1)

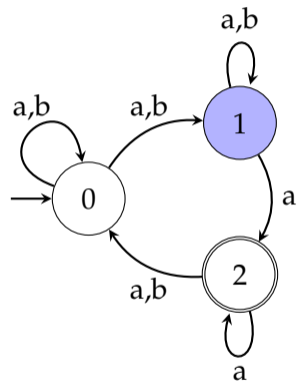
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NFA recognition

as search (with backtracking)



Agenda
(q ₁ , 2)
(q ₁ , 1)

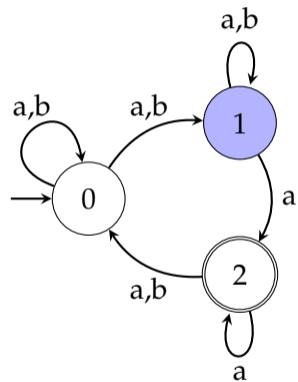
Input:

a	b	a
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NFA recognition

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Agenda
(q ₂ , 3)
(q ₁ , 3)
(q ₁ , 1)

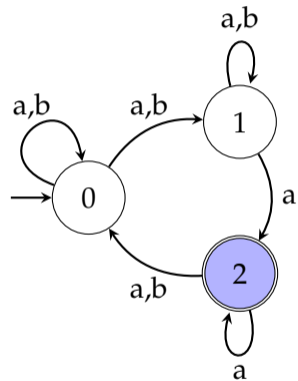
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Input:

a	b	a
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NFA recognition

as search (with backtracking)



Agenda
(q ₂ , 3)
(q ₁ , 3)
(q ₁ , 1)

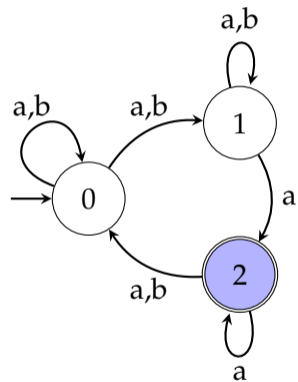
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NFA recognition

as search (with backtracking)



Agenda
(q ₁ , 3)
(q ₁ , 1)

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Input:

a	b	a
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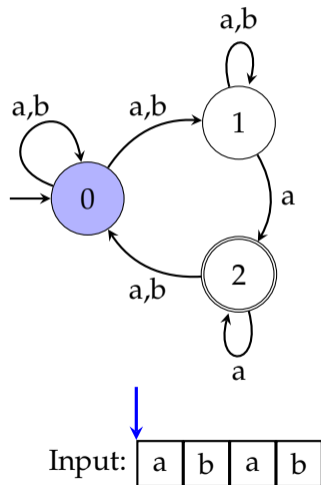
NFA recognition as search

summary

- Worst time complexity is exponential
 - Complexity is worse if we want to enumerate all derivations
- We used a stack as *agenda*, performing a depth-first search
- A queue would result in breadth-first search
- If we have a reasonable heuristic A* search may be an option
- Machine learning methods may also guide finding a fast or the best solution

NFA recognition

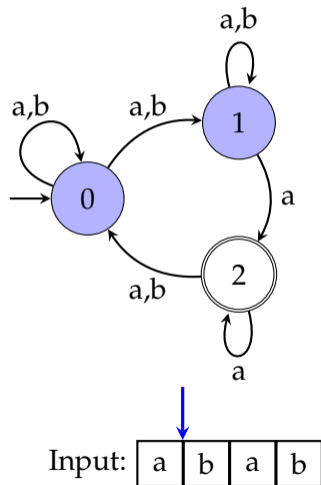
parallel version



1. Start at q_0
2. Take the next input, mark all possible next states
3. If an accepting state is marked at the end of the input, accept

NFA recognition

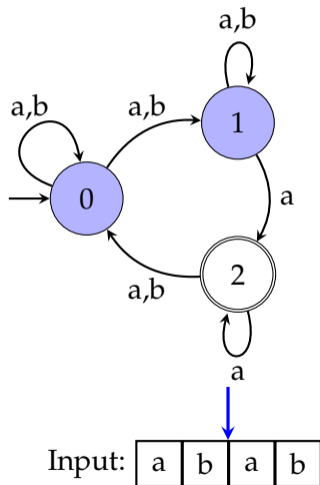
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NFA recognition

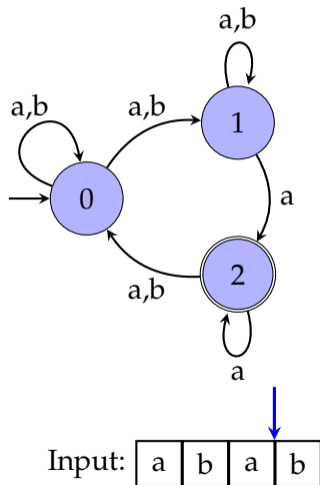
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NFA recognition

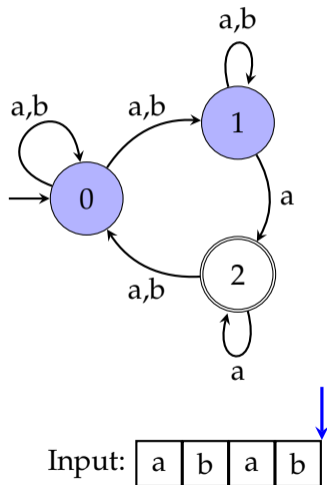
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NFA recognition

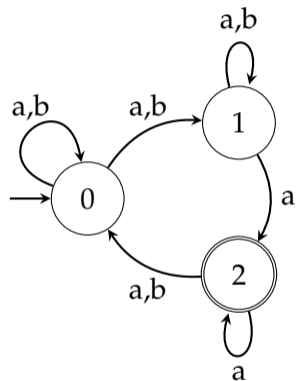
parallel version



1. Start at q_0
2. Take the next input, mark all possible next states
3. If an accepting state is marked at the end of the input, accept

NFA recognition

parallel version



Input:

a	b	a	b
---	---	---	---

1. Start at q_0
2. Take the next input, mark all possible next states
3. If an accepting state is marked at the end of the input, accept

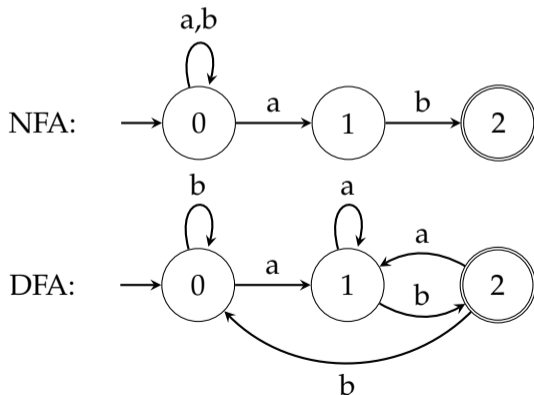
Note: the process is *deterministic*, and *finite-state*.

An exercise

Construct an NFA and a DFA for the language over $\Sigma = \{a, b\}$ where all sentences end with ab .

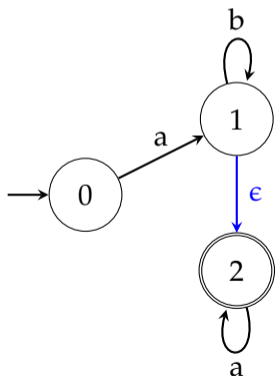
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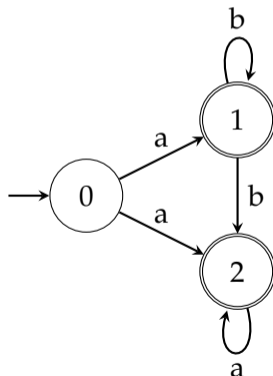
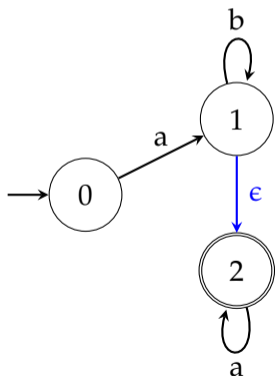
One more complication: ϵ transitions

- An extension of NFA, ϵ -NFA, allows moving without consuming an input symbol, indicated by an ϵ -transition (sometimes called a λ -transition)
- Any ϵ -NFA can be converted to an NFA

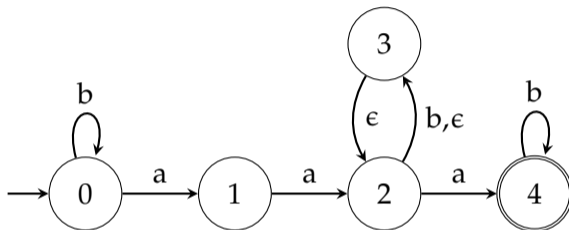


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- An extension of NFA, ϵ -NFA, allows moving without consuming an input symbol, indicated by an ϵ -transition (sometimes called a λ -transition)
- Any ϵ -NFA can be converted to an NFA



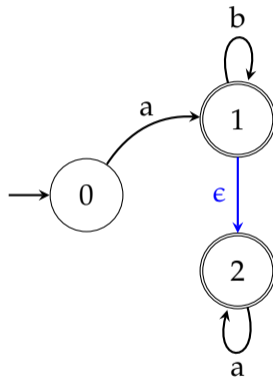
ϵ -transitions need attention



- How does the (depth-first) NFA recognition algorithm we described earlier work on this automaton?
- Can we do without ϵ transitions?

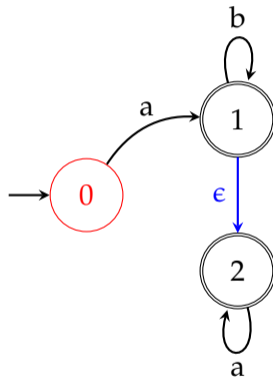
ϵ removal

- We start with finding the ϵ -closure of all states



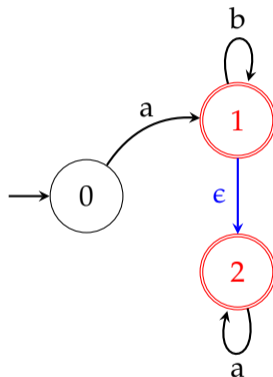
ϵ removal

- We start with finding the ϵ -closure of all states
 - ϵ -closure(q_0) = $\{q_0\}$



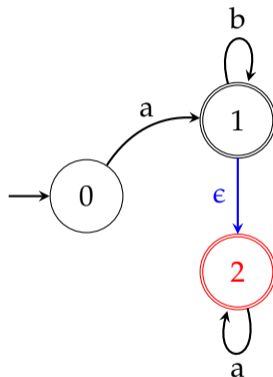
ϵ removal

- We start with finding the ϵ -closure of all states
 - ϵ -closure(q_0) = $\{q_0\}$
 - ϵ -closure(q_1) = $\{q_1, q_2\}$



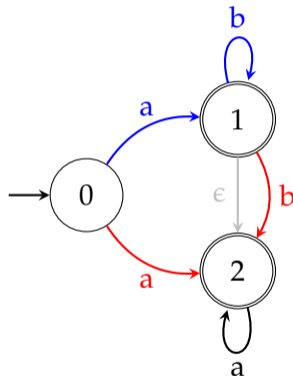
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ϵ removal

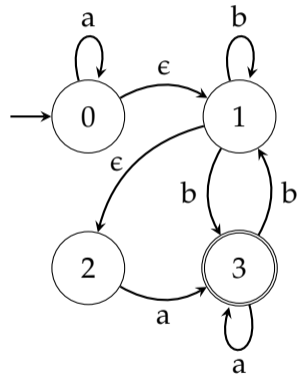
- We start with finding the ϵ -closure of all states
 - ϵ -closure(q_0) = $\{q_0\}$
 - ϵ -closure(q_1) = $\{q_1, q_2\}$
 - ϵ -closure(q_2) = $\{q_2\}$
- Replace each arc to each state with arc(s) to all states in the ϵ -closure of the state



ϵ removal

a(nother) solution with the transition table 1

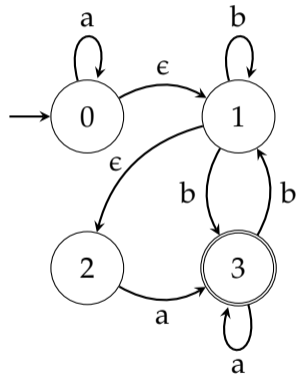
transition table



ϵ removal

a(nother) solution with the transition table 2

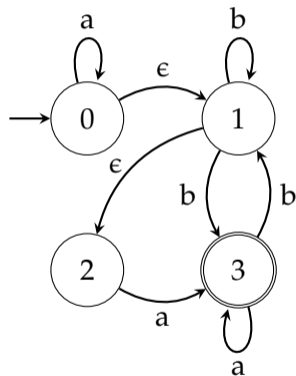
transition table				
		<i>symbol</i>		
		a	b	ϵ
<i>state</i>	\rightarrow 0	0	\emptyset	1
	1	\emptyset	1,3	2
	2	3	\emptyset	\emptyset
	*3	3	1	\emptyset



ϵ removal

a(nother) solution with the transition table 3

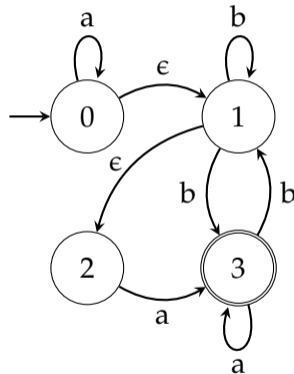
transition table					
		<i>symbol</i>			
		a	b	ϵ	ϵ^*
<i>state</i>	→0	0	\emptyset	1	0,1,2
	1	\emptyset	1,3	2	1,2
	2	3	\emptyset	\emptyset	2
	*3	3	1	\emptyset	3



ϵ removal

a(nother) solution with the transition table 4

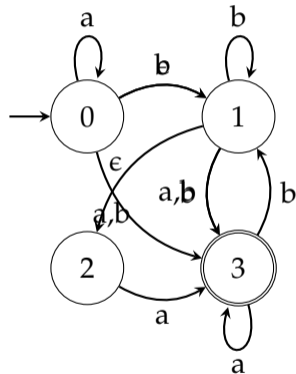
transition table									
		<i>symbol</i>						<i>symbol</i>	
		a	b	ϵ	ϵ^*			a	b
<i>state</i>	\rightarrow 0	0	\emptyset	1	0,1,2	\Rightarrow	\rightarrow 0	0,1,2	1,3
	1	\emptyset	1,3	2	1,2		1	1,2,3	1,2,3
	2	3	\emptyset	\emptyset	2		2	3	\emptyset
	*3	3	1	\emptyset	3		*3	3	1,2



ϵ removal

a(nother) solution with the transition table 5

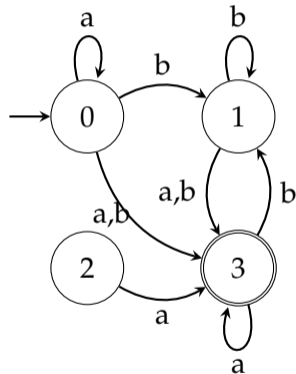
transition table									
		<i>symbol</i>							
		a	b	ϵ	ϵ^*				
<i>state</i>	\rightarrow 0	0	\emptyset	1	0,1,2	\Rightarrow	\rightarrow 0	0,1,2	1,3
	1	\emptyset	1,3	2	1,2		1	1,2,3	1,2,3
	2	3	\emptyset	\emptyset	2		2	3	\emptyset
	*3	3	1	\emptyset	3		*3	3	1,2



ϵ removal

a(nother) solution with the transition table 6

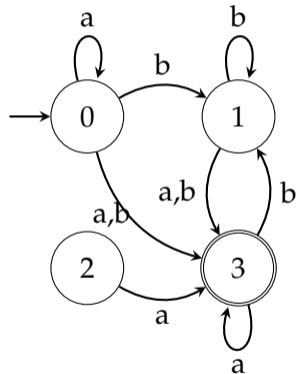
transition table									
		<i>symbol</i>						<i>symbol</i>	
		a	b	ϵ	ϵ^*			a	b
<i>state</i>	\rightarrow 0	0	\emptyset	1	0,1,2	\Rightarrow	\rightarrow 0	0,1,2	1,3
	1	\emptyset	1,3	2	1,2		1	1,2,3	1,2,3
	2	3	\emptyset	\emptyset	2		2	3	\emptyset
	*3	3	1	\emptyset	3		*3	3	1,2



ϵ removal

a(nother) solution with the transition table 7

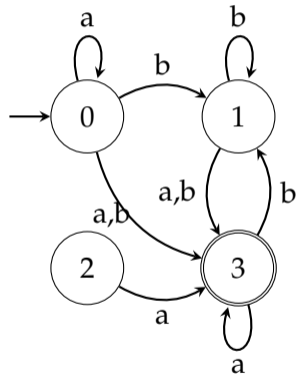
transition table									
		<i>symbol</i>						<i>symbol</i>	
		a	b	ϵ	ϵ^*			a	b
<i>state</i>	\rightarrow 0	0	\emptyset	1	0,1,2	\Rightarrow	\rightarrow 0	0,1,2	1,3
	1	\emptyset	1,3	2	1,2		1	1,2,3	1,2,3
	2	3	\emptyset	\emptyset	2		2	3	\emptyset
	*3	3	1	\emptyset	3		*3	3	1,2



ϵ removal

a(nother) solution with the transition table 8

transition table									
		<i>symbol</i>							
		a	b	ϵ	ϵ^*				
<i>state</i>	\rightarrow 0	0	\emptyset	1	0,1,2	\Rightarrow	\rightarrow 0	0,1,2	1,3
	1	\emptyset	1,3	2	1,2		1	1,2,3	1,2,3
	2	3	\emptyset	\emptyset	2		2	3	\emptyset
	*3	3	1	\emptyset	3		*3	3	1,2



NFA–DFA equivalence

- The language recognized by every NFA is recognized by some DFA
- The set of DFA is a subset of the set of NFA (a DFA is also an NFA)
- The same is true for ϵ -NFA
- All recognize/generate regular languages
- NFA can automatically be converted to the equivalent DFA

Why do we use an NFA then?

- NFA (or ϵ -NFA) are often easier to construct
 - Intuitive for humans (cf. earlier exercise)
 - Some representations are easy to convert to NFA rather than DFA, e.g., regular expressions
- NFA may require less memory (fewer states)

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A quick exercise

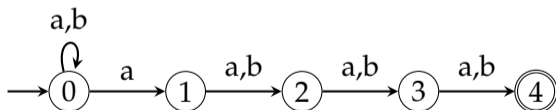
1. Construct (draw) an NFA for the language over $\Sigma = \{a, b\}$, such that 4th symbol from the end is an a

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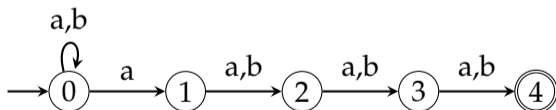


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A quick exercise – and a not-so-quick one

1. Construct (draw) an NFA for the language over $\Sigma = \{a, b\}$, such that 4th symbol from the end is an a



2. Construct a DFA for the same language



Summary

- FSA are efficient tools with many applications
- FSA have two flavors: DFA, NFA (or maybe three: ϵ -NFA)
- DFA recognition is linear, recognition with NFA may require exponential time
- Reading suggestion: Hopcroft and Ullman (1979, Ch. 2&3) (and its successive editions), Jurafsky and Martin (2009, Ch. 2)

Next:

- FSA determinization, minimization
- Reading suggestion: Hopcroft and Ullman (1979, Ch. 2&3) (and its successive editions), Jurafsky and Martin (2009, Ch. 2)

Acknowledgments, credits, references

-  Hopcroft, John E. and Jeffrey D. Ullman (1979). *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley. ISBN: 9780201029888.
-  Jurafsky, Daniel and James H. Martin (2009). *Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics, and Speech Recognition*. second edition. Pearson Prentice Hall. ISBN: 978-0-13-504196-3.

